A Games with Preference for Funding

A PB game with preference for funding (PB/P4F game) is a standard PB game where instead of our standard utility functions we use the extended ones, as defined in Remark 3.1:

$$u'_i(\mathbf{c}) = \begin{cases} \mathbf{c}(p_i) - d(p_i) + \mathbb{1}[\mathbf{c}(p_i) \geq d(p_i)] & \text{if } p_i \in f\big(E(\mathbf{c})\big), \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathbb{1}[x]$ is 1 if x holds and 0 otherwise.

All our NE-(non)existence results for PB games translate to PB/P4F games. Indeed, if a PB game does not admint an equilibrium then neither does its P4F variant (every profitable deviation in a PB game is also profitable in the corresponding P4F game). The other direction is more involved as there are games that have equilibria in our model but do not have them in the P4F variant (see Example A.1). However, if a PB game has an NE where all projects that report values above their delivery costs are funded, then it remains an NE in the corresponding P4F game (a profitable deviation for the P4F game would also be profitable in the original one). All our existence results produce equilibria that satisfy this condition.

Example A.1. *Consider three projects p_1, p_2, p_3, each supported by one distinct voter, with delivery costs $d_1 = 1$, $d_2 = d_3 = 0$, tie-breaking order $p_1 \succ p_2 \succ p_3$ and budget $B = 2$. As we prove in the appendix, the corresponding PB game admits Phragmén-NE $\mathbf{c} = (2, 1, 1)$ but the corresponding P4F PB game has no NEs under Phragmén.*

Below we argue why the statement from the above example indeed holds. Let us denote the PB game from the example as G and the P4F PB game as G'. Using G and G', we show now that in some cases PB games admit a Phragmén-NE, while PB/P4F games (based on the same input) do not.

First, observe that $\mathbf{c} = (2, 1, 1)$ is a Phragmén-NE in PB games. To see that, suppose that it is not and consider a project p_i and a cost c'_i for which $u_i((\mathbf{c}_{-i}, c'_i)) > u_i(\mathbf{c})$. If p_i is p_2 or p_3, we notice that p_i is selected under $\mathbf{c}$ but not under $(\mathbf{c}_{-i}, c'_i)$, for each $c'_i > c_i$. So, neither p_2 nor p_3 can improve its utility. For $p_i = p_1$, we notice that it is not funded under $\mathbf{c}$ and that it can only be funded if $c'_i \leq 1$. But then $u_i((\mathbf{c}_{-i}, c'_i)) \leq 0$. It follows that $\mathbf{c}$ is a Phragmén-NE in PB games.

Second, we show that G' does not admit a Phragmén-NE. For contradiction, suppose that G' admits a Phragmén-NE, say $\mathbf{c}$, for PB/P4F games. As a warm up, we note that at least one project has to be chosen in $E(\mathbf{c})$, as otherwise each project benefits from lowering their cost to $B = 2$.

Further, consider the case that in $\mathbf{c}$ projects have the same cost c. If $c < 1$, then, following the tie-breaking order, p_1 is funded and gets a negative utility. Hence, by increasing its cost to $c' > B > c$, p_1 becomes not funded and increases its utility. Also, if $c \geq 1$, then p_3 is not funded, and as $d_3 = 0$, it can benefit from decreasing its cost.

Now, suppose that in $\mathbf{c}$ projects do not have the same costs. Let $c_1 \geq 1$. Then, if $c_2 \geq c_1$ or $c_3 \geq c_1$, then some project that is not funded would benefit from lowering its cost. For a complementary case of $c_2 < c_1$ and $c_3 < c_1$, we further consider two subcases. If $c_2 + c_3 < 2$, then one of p_2 and p_3 would benefit from increasing their cost. Alternatively, $c_2 + c_3 \geq 2$. This implies that $c_1 > 1$ and that $c_2 \geq 1$ or $c_3 \geq 1$. In this case p_1 would benefit, in a PB/P4F game, from choosing $c'_1 = 1$ and becoming selected.

Finally, if $c_1 < 1$, then it benefits from increasing its cost if it is funded under $\mathbf{c}$. Notice further that if it would not be funded, then $c_2 < 1$ and $c_3 < 1$, and both of these costs are lower than c_1. It follows that one of them can increase its utility by increasing its cost.

As our case distinction is exhaustive, we know that there is no Phragmén-NE for G'. Consequently, we proved our claim about the equilibria in games G and G'.

B Missing Proofs

B.1 Proof of Proposition 4.1

Proof. Take a PB game and a strategy profile $\mathbf{c}$ as described in the statement, and let p be the project such that $\mathbf{c}(p) = B$. If a project other than p increases its cost, it still is not selected so this is not a beneficial move. If p decreases its cost then its utility drops, and if it increases its cost then it is not funded. In either case, its utility decreases. So, $\mathbf{c}$ is a BasicAV-NE. $\square$

B.2 Proof of Proposition 4.2

In our proof we first use the following proposition.

Proposition B.1. *Take a PB game and the corresponding strategy profile* $\mathbf{ap}$*. If* $d(p) \leq \mathbf{ap}(p)$ *for each project* p*, then if a profile* $\mathbf{c}$ *is an AV/Cost-NE for this game, then (1)* $\sum_{p \in P} \mathbf{c}(p) = B$*, and (2) AV/Cost funds all the projects.*

Proof. Let the notation be as in the statement of the proposition. In the beginning we observe that if $\sum_{p \in P} \mathbf{c}(p) < B$, then AV/Cost selects all the projects. Hence, the project considered last benefits by reporting a higher cost (so that the sum of the reported costs is B). Thus such a strategy profile $\mathbf{c}$ is not an equilibrium.

Next, let us assume that $\sum_{p \in P} \mathbf{c}(p) > B$. Let P_{won} and P_{lost} be sets of projects that, respectively, are and are not funded. By our assumption, we know that P_{lost} must be nonempty, and we also note that P_{won} must be nonempty. Naturally, $\sum_{p \in P_{won}} \mathbf{c}(p) \leq B$. For each two projects p_i and p_j in P_{won}, it must be the case that $|A(p_i)|/\mathbf{c}(p_i) = |A(p_j)|/\mathbf{c}(p_j)$. For example, if we had $|A(p_i)|/\mathbf{c}(p_i) > |A(p_j)|/\mathbf{c}(p_j)$ then AV/Cost would consider p_i prior to p_j and, consequently, it would be beneficial for p_i to increase its reported cost by a small-enough amount so that it would still be considered (and selected) prior to p_j. This would contradict the assumption that $\mathbf{c}$ is an equilibrium. Next, since $\sum_{p \in P} \mathbf{ap}(p) = B$, there must be some project p' in P_{won} such that $\mathbf{c}(p') > \mathbf{ap}(p') \geq d(p')$ (otherwise, if each project $p' \in P_{won}$ reported at most $\mathbf{ap}(p')$, then any project $p \in P_{lost}$ would be selected after reducing its cost to $\mathbf{ap}(p)$, so $\mathbf{c}$ could not have been NE). Further, by definition of the approval-proportional profile, for every project $p \in P$, we have that $|A(p)|/\mathbf{ap}(p)$ is the same value and, so, for every project $q \in P_{lost}$ it holds that $|A(q)|/\mathbf{ap}(q) = |A(p')|/\mathbf{ap}(p') > |A(p')|/\mathbf{c}(p')$. This means that every project $q \in P_{lost}$ can improve its utility by reporting cost $\mathbf{ap}(q)$ and being selected. This contradicts the assumption that $\mathbf{c}$ is an equilibrium.

Thus it must be the case that $\sum_{p \in P} \mathbf{c}(p) = B$, which implies that AV/Cost selects all projects. $\square$

Proof of Proposition 4.2. Consider a PB game (P, V, B, d) such that for every $p_i \in P$ it is true that $d(p) \leq \mathbf{ap}(p)$.

We argue that strategy $\mathbf{c} = \mathbf{ap}$ is a AV/Cost-NE. Note that the sum of the reported costs in $\mathbf{c}$ is exactly B and all projects are funded. Hence, for each player, decreasing the cost leads to a utility loss. Hence, a profitable deviation could only be through increasing the reported cost. Towards contradiction, let us fix some player $p_i \in \mathbf{ap}$ and assume that reporting a cost $c'_i > c_i$ leads to a better payoff. This means that p_i is funded in the modified election $E' = (P, V, B, (\mathbf{c}_{-i}, c'_i))$, that is, $p_i \in \mathrm{AV/cost}(E')$. Additionally, since by Proposition B.1 we know that $\sum_{j \in |P|} c_j = B$, it immediately follows that $c'_i + \sum_{j \in |P| \setminus \{i\}} c_j > B > \sum_{j \in |P| \setminus \{i\}} c_j$. It is also the case that for each $j \in |P|$, $|A(P_j)|/c_j > |A(P_i)|/c'_i$. Hence, in particular, in the run of $\mathrm{AV/cost}$ for E', all projects are considered before p_i. Since all of these project are selected, by the time the procedure starts considering p_i, the remaining budget is smaller than c'_i, so $p_i \notin \mathrm{AV/cost}(E')$; a contradiction. Note that because all projects that are funded are always considered by the procedure in the same time, the result is independent of the tie-breaking order $\succ$.

We now argue that every strategy profile $\mathbf{c}$ other than $\mathbf{ap}$ is not a $\mathrm{AV/cost}$-NE. To this end, we take such a profile $\mathbf{c}$ and assume towards contradiction that it is a $\mathrm{AV/cost}$-NE. Observe that since $\mathbf{c} \neq \mathbf{ap}$, there exist two distinct projects p_i and p_j such that $|A(p_i)|/c_i \neq |A(p_j)|/c_j$. Hence one of the projects is considered earlier than the other; we assume without loss of generality that p_i is considered before p_j. Notably, as $\mathbf{c}$ is (by our assumption) a $\mathrm{AV/cost}$-NE, by Proposition B.1, we know that $\{p_i, p_j\} \subseteq \mathrm{AV/cost}(\mathbf{c})$. This implies, however, that p_i can report a slightly higher price c'_i and still be selected by $\mathrm{AV/cost}$ thus obtaining a better payoff. Formally, there exists a $c'_i > c_i$ resulting in the profile $(c'_i, \mathbf{c}_{-i})$ in which $|A(P_i)|/c'_i < |A(P_j)|/c_j$. So, p_i is (still) considered by $\mathrm{AV/cost}$ before c_j and the deviation is small enough (that is, $c'_i - c_i < c_j$) to guarantee that $p_i \in \mathrm{AV/cost}((\mathbf{c}_{-i}, c'_i))$. So, $\mathbf{c}$ is not a $\mathrm{AV/cost}$-NE; a contradiction. Again, due to Proposition B.1,

Data: set $P = \{p_1, p_2, \ldots, p_m\}$ of projects with the delivery costs function d, set V of voters with their ballots, budget B, A/D tie-breaking $\succ$.
Result: profile $\mathbf{c}$ that is a $\mathrm{AV/cost}$-NE under tie-breaking $\succ$.

1: $B^* \leftarrow B$ // remaining budget
2: $\mathbf{c} \leftarrow (d(p_1), d(p_2), \ldots, d(p_m)) = \mathbf{c}_0$ // initial strategy
// let $t(p)$, for each $p \in P$, be $d(p)/A(p)$
// note that $\succ$ is nondecreasing w.r.t. values of t
// $\mathrm{pos}_X(i)$ denotes the top ith project of order $\succ$ restricted to $X \subseteq P$
3: $P_\mathrm{p} \leftarrow \{p \in P : c(p) \leq B^*\}$ // prospective projects
4: **while** $P_\mathrm{p} \neq \emptyset$ **do** // are there projects to consider?
5: $\quad P' \leftarrow P_\mathrm{p}$ // P' is a helper variable
6: $\quad d(\mathrm{pos}_{P'}(0)) \leftarrow -\inf$ // guardian "fake" value
7: $\quad d(\mathrm{pos}_{P'}(|P'|+1)) \leftarrow +\inf$ // guardian "fake" value
8: $\quad k \leftarrow$ maximum integer $x \leq |P'|$ such that
$\quad\quad t(\mathrm{pos}_{P'}(x)) \sum_{i \leq x-1} A(\mathrm{pos}_{P'}(i)) \leq B^* < t(\mathrm{pos}_{P'}(x+1)) \sum_{i \leq x+1} A(\mathrm{pos}_{P'}(i))$
9: $\quad T \leftarrow$ maximum $x \leq d(\mathrm{pos}_{P'}(k+1))/|A(\mathrm{pos}_{P'}(k+1))|$
$\quad\quad$ such that $x \sum_{i \in [k]} A(\mathrm{pos}_{P'}(i)) \leq B^*$
10: $\quad$ **for** $i = 1$ **to** k **do** // update $\mathbf{c}$ and B^*
11: $\quad\quad c(\mathrm{pos}_{P'}(i)) \leftarrow T \cdot A(\mathrm{pos}_{P'}(i))$
12: $\quad\quad B^* \leftarrow B^* - c(\mathrm{pos}_{P'}(i))$
13: $\quad\quad P_\mathrm{p} \leftarrow P_\mathrm{p} \setminus \{\mathrm{pos}_{P'}(i)\}$
14: $\quad$ **end for**
15: $\quad$ **if** $k < |P'|$ **then**
16: $\quad\quad P_\mathrm{p} \leftarrow P_\mathrm{p} \setminus \{\mathrm{pos}_{P'}(k+1)\}$
17: $\quad$ **end if**
18: $\quad P_\mathrm{p} \leftarrow \{p \in P_\mathrm{p} : c(p) \leq B^*\}$
19: **end while**
20: **return** $\mathbf{c}$

Algorithm 1: Finding a Nash equilibrium for $\mathrm{AV/cost}$.

in each Nash equilibrium, all projects are selected to be funded. Hence, our proof works for every possible tie-breaking. □

B.3 Proof of Theorem 4.4

Proof. Our Algorithm 1 computes the claimed AV/Cost-NE.

Let us fix a PB game $G = (P, V, B, d)$ and some corresponding A/D tie-breaking $\succ$, as specified in the theorem statement. We first introduce helpful notation and discuss $\succ$ in more detail. Then, we proceed with presenting a high-level description of Algorithm 1 that finds the claimed AV/Cost-NE, which we refer to as $\mathbf{c}$. Eventually, we prove the correctness of the algorithm using Claim 1 and conclude with proving Claim 1 itself.

Recall that by definition $\succ$ orders the projects p_i nonincreasingly according to approval-to-delivery-cost ratios $|A(p_i)|/d(p_i)$. Crucially, $\succ$ is always compatible with the order in which AV/Cost considers the projects assuming they report their delivery costs, that is, where for all $p_i \in P$, $c(p_i) = d(p_i)$. In what follows, for each set of projects $X \subseteq P$, each $p_i \in X$, and the corresponding suborder $\succ_X$ of $\succ$, we write $\mathrm{pos}_X(p_i)$ to denote the position of p_i in $\succ_X$. Analogously, for some natural number $x \leq |X|$, we denote by $\mathrm{top}_X(x)$ the set of top x projects according to $\succ_X$.

Algorithm 1 constructs the AV/Cost $\mathbf{c}$ iteratively, starting from strategy $\mathbf{c}_0$ in which each project reports its delivery cost (Line 2). In each iteration of the while loop, the algorithm selects a group of projects that "underreport" their costs compared to the support that they get. As the next step, the algorithm increases the reported costs of these projects in strategy AV/Cost as much as possible ensuring that the projects remain funded. If the funded projects after this update do not exhaust the budget, the procedure is repeated for the remaining projects, whose prices have not been increased so far. Such projects, albeit worse regarding the delivery-cost-to-support ratio, can still benefit from the increase. The algorithm outputs $\mathbf{c}$ if one of the updates step lead to the situation in which the whole budget is used or when there is no more projects underreporting their costs.

In the following more detailed description of Algorithm 1, we use the notation as specified in the pseudocode and whenever the value k in an iteration of the loop is smaller than $|P'|$, we define $p^* = \mathrm{pos}_{P'}(k+1)$ (assuming the value of P' from the corresponding iteration). Central to Algorithm 1 is the while loop. Importantly, due to our basic assumption on the delivery costs of the projects (i.e., that for each $p \in P$, $d(p) \leq B$), initially set P_p is not empty (Line 3), so the while loop runs at least once. Line 9 of the algorithm guarantees that in each iteration the loop sets the values of T and k to:

$$k < |P'| \text{ and } T = {}^{d(p^*)}/_{A(p^*)} \text{ or} \tag{S1}$$

$$k < |P'| \text{ and } T \cdot \sum_{p_j \in \mathrm{top}_{P'}(k)} |A(p_j)| = B^*, \text{ or} \tag{S2}$$

$$k = |P'| \text{ and } T \cdot \sum_{p_j \in \mathrm{top}_{P'}(k)} |A(p_j)| = B^*. \tag{S3}$$

The aforementioned case distinction is crucial for the following claim.

Claim 1. *Right after executing Line 18 in each iteration of the while loop of Algorithm 1, it jointly holds that:*

1. *all projects from $\mathrm{top}_{P'}(k)$ are funded under strategy profile $\mathbf{c}$ and none of them has a profitable deviation for $\mathbf{c}$;*
2. *either $k = |P'|$ or there exists a project $p^* = \mathrm{pos}_{P'}(k+1)$ that is not funded and has no profitable deviation for profile $\mathbf{c}$;*
3. *no project from $\mathrm{top}_{P'}(k)$ has a profitable deviation for each strategy profile $\mathbf{c}'$ that differs from $\mathbf{c}$ only by strategies of projects in P_p in a way that, for each $p \in P_P$, $\mathbf{c}'(p) \geq \mathbf{c}(p)$;*
4. *either $k = |P'|$ or the project $p^* = \mathrm{pos}_{P'}(k+1)$ has no profitable deviation for each strategy profile $\mathbf{c}'$ that differs from $\mathbf{c}$ only on strategies of projects in P_p such that, for each $p \in P_P$, $\mathbf{c}'(p) \geq \mathbf{c}(p)$;*
5. *no project in $P \setminus P_p$ has a profitable deviation in $\mathbf{c}$.*

Note that Algorithm 1 ends when executing Line 18 makes P_p empty. So, by Item 5 of Claim 1 the algorithm returns profile $\mathbf{c}$ which is a Nash equilibirum. Clearly, in each iteration $k > 0$, so at least one element is removed from P_p in every iteration of the loop. Consequently, the algorithm always ends and thus it is correct. To conclude the whole proof it remains to show that Claim 1 indeed holds.

Proof of Claim 1. We provide an inductive argument over the iterations of the while loop of Algorithm 1. For the sake of the argument's simplicity, instead of thinking of AV/Cost considering projects in nonincreasing order of approval-to-cost ratio, we say that it considers projects in nondecreasing order of cost-to-approval ratio. We note that these two interpretations are equivalent. We take the first iteration of the loop as the base case and subsequently show that Items 1 to 5 hold.

Item 1. We first show that each $p \in \mathrm{top}_{P'}(k)$ is funded in election $E(\mathbf{c})$. Let us denote as $t(p)$ the ratio ${}^{\mathbf{c}(p)}/_{|A(p)|}$. Due to Line 9 and the following foreach loop, we know that every project in $\mathrm{top}_{P'}(k)$ is tied to be considered by AV/Cost due to the same cost-to-approval ratio T. So, if $k = |P'|$, then all projects in $P_\mathrm{p} = P$ are tied for consideration. Since, by Line 9 all project costs sum up to $B^* = B$, the claim holds. We follow assuming that $k < |P'|$. In this case, Line 9 shows that p^* has cost-to-approval ratio $t(p^*) \geq T$. Recall that $\succ$ orders projects nondecreasingly with respect to their cost-to-approval ratio assuming $\mathbf{c}_0$ and that $\mathbf{c}$ differs from $\mathbf{c}_0$ only in strategies of projects in $\mathrm{top}_{P'}(k)$. Hence, p^* precedes in $\mathbf{c}$ each project except of those in $\mathrm{top}_{P'}(k)$ and all projects in $\mathrm{top}_{P'}(k)$ are considered before p^* (the latter holds even in the case of $t(p^*) = T$ due to $\succ$). Eventually, Line 9 guarantees that the cost of all projects in $\mathrm{top}_{P'}(k)$ does not exceed the budget. Knowing that, we show that no project in $\mathrm{top}_{P'}(k)$ has a profitable deviation in $\mathbf{c}$. To prove it by contradiction, let us assume that some player $p_i \in \mathrm{top}_{P'}(k)$ has a profitable deviation by reporting cost $c'_i > c_i$; the opposite case of $c'_i < c_i$ trivially does not yield a payoff improvement for p_i. As a result, p_i has to be funded in the corresponding election $E((\mathbf{c}_{-i}, c'_i))$. In this election, p_i has cost-to-approval ratio $t' = c'_i/|A(p_i)| > T$. We further split our analysis into the three cases from Equations (S1) to (S3).

Assuming Equation (S1), project p^* is also funded in election E', as it is considered before p_i due to $t' > T$. Hence, the funded projects cost at least

$$T \cdot \sum_{\substack{p_j \in \mathrm{top}_{P'}(k) \\ p_j \neq p_i}} |A(p_j)| + t'|A(p_i)| + T|A(p^*)| > \\ T \cdot \sum_{p_j \in \mathrm{top}_{P'}(k+1)} |A(p_j)|.$$

However, due to Line 8, we know that $B^* = B < T \sum_{p_j \in \mathrm{top}_{P'}(k+1)} |A(p_j)|$, which gives the contradiction with the fact that p_i is funded.

Suppose Equation (S2) or Equation (S3) hold. Then, given our assumption that p_i is funded, the total cost of funded projects is

$$T \cdot \sum_{\substack{p_j \in \mathrm{top}_{P'}(k) \\ p_j \neq p_i}} |A(p_j)| + t'|A(p_i)| > \\ T \sum_{p_j \in \mathrm{top}_{P'}(k)} |A(p_j)| = B^* = B,$$

Here, the equality is due to the assumption of Equation (S2) or Equation (S3); which yields the sought contradiction.

Item 2. If $k = |P'|$, then the statement trivially holds. Otherwise, let us consider p^* in the light of Equation (S1). Due to Line 8, we have that

$$B < T \sum_{p_j \in \mathrm{top}_{P'}(k+1)} |A(p_j)| = \\ T \sum_{p_j \in \mathrm{top}_{P'}(k)} |(A(p_j))| + d(p^*).$$

Hence, since all projects in $\mathrm{top}_{P'}(k)$ are funded, p^* is not funded. Naturally, if p^* reports a cost bigger than $d(p^*)$ it will not be funded even more, whereas reporting a cost lower than $d(p^*)$ results in a worse payoff, which finishes the argument for, for Equation (S1). By the condition of Equation (S2), it immediately holds that all projects $\mathrm{top}_{P'}(k)$ use up the whole budget. As a result, again, p^* is not funded and, analogously to the case of Equation (S1), p^* has no profitable deviation. We already observed that Item 2 trivially holds for $k = |P'|$, which subsumes Equation (S3).

Items 3 and 4. Observe that assuming strategy $\mathbf{c}$, all projects in P_p are considered after all projects outside of P_p in election $E(\mathbf{c})$. This is because order $\succ$ is compliant with the order of considering the projects by AV/Cost and our modifications to the initial strategy profile $\mathbf{c}$ do not change the order of considering projects (note that we let modified projects be tied for consideration with cost-to-approval ratio T). Furthermore, note that Algorithm 1 never decreases the reported costs of the projects and never considers the same project twice, so the projects in P_p will never be considered before the other ones as a result of further modifications of $\mathbf{c}$. So, no modification of the profile $\mathbf{c}$ that increases the costs for projects in P_p can influence the decision made for projects in $\mathrm{top}_{P'}(k)$ or $\mathrm{top}_{P'}(k+1)$ (depending on whether $k < |P'|$), as the latter are considered by AV/Cost earlier than projects in P_p.

Item 5. Note that $P \setminus P_\mathrm{p}$ consists only of the following projects: (i) those removed in the foreach loop, that is, $\mathrm{top}_{P'}(k)$; (ii) p^* if $k < |P'|$; and (iii) those removed in Line 18. Regarding the first two groups, we have already shown that they do not have a profitable deviation for $\mathbf{c}$. Let $\hat{p}$ be $\mathrm{top}_{P'}(k)$ if $k < |P'|$ or, otherwise, let $\hat{p}$ be p^*. The last group Y consists of these projects $p \in P'$, for which $\hat{p} \succ p$ and whose $d(p) > B^*$. Hence, by $\succ$, these projects are considered after all projects in $\mathrm{top}_{P'}(k) \cup \{\hat{p}\}$. However, projects in $\mathrm{top}_{P'}(k)$ are selected to be funded, which leaves exactly B^* remaining budget. So there is not enough budget left to fund any project in Y, even if it reports its delivery cost. Hence, no project in Y has a profitable deviation.

Thus, we have established the base case and we move on to the induction step. Consider $x > 1$ and the x-th iteration of the while loop, assuming that the base case claims are met for the $(x-1)$-th iteration. Due to the assumptions for the $(x-1)$-th iteration together with the fact that our algorithm never changes the order $\succ$ in which AV/Cost considers projects and that it never considers the same project twice, we can ignore all projects processed in previous $(x-1)$ iterations of the while loop. Clearly, the ignored projects will have no impact on the current loop iteration except for decreasing the value of the variable B^* representing the remaining budget. Consequently, the arguments for Items 1 to 4 carry on without changes for the xth iteration of the while loop. The claim from Item 5, however, needs more attention. Let $Q^{(x-1)} = P \setminus P_{\mathrm{p}}^{(x-1)}$ be the set of players without profitable deviations after the $(x-1)$-th iteration of the loop. Note that $P' = P_{\mathrm{p}}^{(x-1)}$. We denote by R the set of projects, that are removed from the initial state of P_{p} by Lines 13 and 18. That is, R consists of all these projecst that—due to Items 1 to 4 and the argument for Item 5 in the base case—have no profitable deviation in $\mathbf{c}$. So, we have that $P_{\mathrm{p}} = P' \setminus R$. We now consider the set $Q = P \setminus P_{\mathrm{p}}$ of the projects for which we need to show that they have no profitable deviation. Putting the bits together, we observe the following:

$$\begin{aligned} Q =& P \setminus P_{\mathrm{p}} = P \setminus (P' \setminus R) = \\ & P \setminus (P_{\mathrm{p}}^{(x-1)} \setminus R) = (P \cap R) \cup (P \setminus P_{\mathrm{p}}^{(x-1)}). \end{aligned}$$

Since $P \cap R = R$, we obtained that $Q = R \cup Q^{(x-1)}$ thus showing that the x-th iteration extends the set of project that do not have a profitable deviation with a collection of project that have no profitable deviation either. As a result, we proved Item 5 for the x-th iteration. ■

Having proven Claim 1, we completed the argument for the correctness of Algorithm 1 and Theorem 4.4. □

B.4 Proof of Proposition 4.5

Proof. Take a natural $\gamma > 1$ and consider a PB game (P, V, B, d) where $P = \{p_1, p_2, p_3\}$, $B = 10$, $d(p_1) = d(p_2) = 0$, and $d(p_3) = 10 - \frac{1}{2\gamma}$. The voters have plurality ballots such that $|A(p_1)| = |A(p_2)| = 1$ and $A(p_3) = 20\gamma - 1$. We assume tie-breaking order $p_1 \succ p_2 \succ p_3$. We claim that strategy profile $\mathbf{c}$ such that $\mathbf{c}(p_1) = \mathbf{c}(p_2) = \frac{1}{2\gamma}$ and $\mathbf{c}(p_3) = 10 - \frac{1}{2\gamma}$ is a Phragmén-NE.

First, we see that if the projects reports costs as in $\mathbf{c}$ then Phragmén selects p_1 and p_2. Indeed, we see that at time moment $\frac{1}{2\gamma}$ the voters supporting each of the projects have exactly as much money as is need to purchase them. Due to tie-breaking, the singleton voters supporting p_1 and p_2 buy these projects and, then, there is not enough budget left for p_3 and the rule finishes.

Second, we observe that no project can benefit by changing its strategy under $\mathbf{c}$. Indeed, if either p_1 or p_2 decreased their cost, they would obtain lower payoff, and if either of them increased their cost, p_3 would be funded instead and the payoff of the project that increased its cost would drop to zero. If p_3 decreased its cost then it would be selected, but its payoff would become negative (due to the delivery cost), and if it increased its cost then its payoff would remain zero. Consequently, $\mathbf{c}$ is Phragmén-NE.

Finally, we have $\mathbf{c}(p_1) + \mathbf{c}(c_2) = {}^{1}/_{\gamma}$, and as $B = 10$, this is less than ${}^{B}/_{\gamma}$. This concludes the proof. □

B.5 Proof of Proposition 4.7

Proof. Consider a PB game G and the strategy profile $\mathbf{c}$ as defined in the statement of the proposition. We claim that $\mathbf{c}$ is a Phragmén-NE. First, we observe that under $\mathbf{c}$ Phragmén selects all the projects, so it is not beneficial for any of them to report a lower cost. On the other hand, if some project p reports a higher cost, this project is not selected by Phragmén. To see why this is the case, consider some project p. Under $\mathbf{c}$, the total cost of the projects in $party(p)$ is ${}^{B \cdot |A(p)|}/_{|V|}$. Since each of the $|A(p)|$ voters supporting these projects earns one unit of money per one unit of time, altogether they earn this money in time $B/|V|$. This value is independent of p so, under $\mathbf{c}$, the last project of each party is funded at the same time. Hence, if some project increased its price, it would be considered even later and, by that time, there would not be enough budget left. Hence, it is never beneficial to increase a cost and so, $\mathbf{c}$ is a Phragmén-NE.

It remains to show that $\mathbf{c}$ is the unique Phragmén-NE for our game. To this end, let $\mathbf{c}'$ be some arbitrary equilibrium for G. By Proposition B.1, we know that under $\mathbf{c}'$ the reported costs of all projects sum up to B and all projects are funded. Next, we observe that for each project p, all projects from $party(p)$ report the same cost. Indeed, if there were two projects, p' and $p'' \in party(p)$, such that $\mathbf{c}'(p') < \mathbf{c}'(p'')$, then it would be beneficial for p' to report a higher cost (but below $\mathbf{c}'(p'')$), so that Phragmén would still consider and fund it prior to p'' (for which, then, there would not be enough budget left).

Finally, if there are two projects, p and q, such that $party(p) \neq party(q)$, then, under $\mathbf{c}'$, the last project from $party(p)$ and the last project from $party(q)$ are selected by Phragmén at the same time. Indeed, if this were not the case, then it would be beneficial for the one selected earlier to report higher cost (but so that it still is selected at an earlier time than the other project).

Altogether, the only strategy profile that satisfies the properties described above is $\mathbf{c}$ as defined in the statement of the proposition. ☐

B.6 Proof of Proposition 4.8

Proof. Let our project set be $P = P' \cup P''$, where $P' = \{p'_1, p'_2, p'_3\}$ and $P'' = \{p''_1, p''_2, p''_3\}$. Similarly, the set of voters is $V = V' \cup V''$, where $V' = \{v_{1'}, v_{2'}, v_{3'}\}$ and $V'' = \{v_{1''}, v_{2''}, v_{3''}\}$. The approvals are as follows (see also Figure 3 for illustration):

1. p'_1 is approved by $v_{1'}$, p'_2 is approved by $v_{2'}$, and p'_3 is approved by all the voters in V'.
2. The approvals for the projects in P'' are analogous, except that they are approved by the voters from V''.

We set the delivery costs d to be zero for every project, and we set the budget B to be 1 (the exact value will be irrelevant as we will operate on times when Phragmén reaches the costs of particular projects rather than on directly on these costs). We claim that under Phragmén there are no Nash equilibria for the thus defined PB game $G = (P, V, B, d)$.

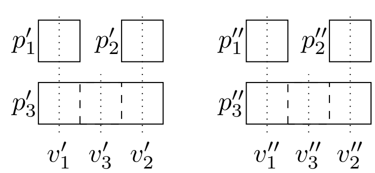

Figure 3: Illustration of the PB game from the proof of Proposition 4.8. The projects are depicted as boxes. Each voter approves those projects that are drawn directly above him or her (and are crossed by the dotted line).

For the sake of contradiction, let us assume that there is Phragmén-NE for G and let it be $\mathbf{c}^*$. For each project $p \in P$, let $time(p)$ be the time moment (in the sense of the Phragmén rule) when the voters approving p would collect exactly $\mathbf{c}^*(p)$ amount of money (assuming that neither of these voters spends it on any other projects in between; hence $time(p) = \mathbf{c}^*(p)/|A(p)|$). We make the following observations:

1. All projects in P are funded for the reported costs $\mathbf{c}^*$ (if some project were not funded, it would prefer to report a small nonzero cost that would be at most equal to B and that would ensure that Phragmén considers it first).
2. It must be the case that $time(p'_1) = time(p'_2)$. Indeed, if we had $time(p'_1) < time(p'_2)$ then it would be beneficial for p'_1 to slightly increase its cost, but so that $time(p'_1)$ would still be below p'_2. Then, irrespective in what order are the other projects funded, p'_1's voter would collect enough money to purchase p'_1 before the p'_2's voter would, and p'_1 would be

bought (since p'_2 would not be selected at this time yet, there would be enough budget left for this). This would contradict that $\mathbf{c}^*$ is an equilibrium. The case $time(p'_2) < time(p'_1)$ is symmetric.

3. It must be the case that $time(p'_3) = time(p'_1)$ and, hence, also equal to $time(p'_2)$. Indeed, if we had $time(p'_3) < time(p'_1) = time(p'_2)$ then it would be beneficial for p'_3 to report slightly higher cost, so that $time(p'_3)$ would still be smaller than $time(p'_1) = time(p'_2)$, yet p'_3's cost would not have increased by more than the total costs of p'_1 and p'_2. Then p'_3 would still be selected before p'_1 and p'_2 and there would be sufficient amount of budget left for it. This would contradict that $\mathbf{c}^*$ is an equilibrium. On the other hand, if $time(p'_3) > time(p'_1) = time(p'_2)$ then it would be beneficial, e.g., for p'_1 to report slightly higher cost, but ensuring that $time(p'_1) < time(p'_3)$. Indeed, if originally we have $time(p'_1) < time(p'_3)$, then p'_1 is funded before p'_3 by Phragmén. After the increase, p'_1 would still be purchased before p'_3 and, thus, there would still be sufficient budget for it. This would contradict that $\mathbf{c}^*$ is an equilibrium.

4. By reasoning analogous to the one given above, it must be the case that $time(p''_1) = time(p''_2) = time(p''_3)$.

Given the above observations, we set:

$$time' = time(p'_1) = time(p'_2) = time(p'_3), \text{ and}$$
$$time'' = time(p''_1) = time(p''_2) = time(p''_3).$$

Note that this implies that $\mathbf{c}^*(p'_1) = \mathbf{c}^*(p'_2) = time'$, $\mathbf{c}^*(p''_1) = \mathbf{c}^*(p''_2) = time''$, $\mathbf{c}^*(p'_3) = 3 \cdot time'$, and $\mathbf{c}^*(p''_3) = 3 \cdot time''$. We claim that $time' = time''$. Indeed, let us consider what happens if $time' < time''$ (the case where $time'' < time'$ is symmetric). There are two cases two consider (the second case also splits into two).

Case A First, let us assume that at least one of p'_1 and p'_2 is preferred to p'_3 by the tie-breaking order (for the sake of specificity, let this be p'_1). Then it is beneficial for p'_2 to slightly increase its cost. To see why this is the case, let us consider how Phragmén operates after this increase. At time $time(p'_1) = time(p'_3)$, the voters have enough funds to purchase either p'_1 or p'_3 (and not enough to purchase p'_2, who increased its cost). The rule selects p'_1 due to tie-breaking. Consequently, voter $1'$ pays for p'_1 and his or her virtual bank account is reset to zero. Voters $2'$ and $3'$ still have $time'$ amount of money. The cost of p'_3 is $3 \cdot time'$, so voters in N' will have collected enough money for it after further $1/3 time'$ amount of time. However, if p'_2 increased its cost from $time'$ to an amount a bit below $4/3 \cdot time'$ then its voter will have collected this amount earlier, and p'_2 will be purchased before p'_3. This contradicts the fact that $\mathbf{c}^*$ is an equilibrium.

Case B′ The second case is that p'_3 is preferred by tie-breaking to both p'_1 and p'_2. Then it is beneficial for p'_3 to slightly increase its cost. Again, let us consider how Phragmén operates after such a change. If p''_3 is selected prior to both p''_1 and p''_2, then at time $time''$ (after the purchase of p''_3) the voters have the following amounts of money:

1. Voter $3'$ has $time''$ amount of money and the remaining voters in N' have nonzero amounts of money (specifically, each of them has $time'' - time'$, because they paid for p'_1 and p'_2 at time $time'$).

2. All the voters in V'' have empty bank accounts.

Hence, only after another $time''$ amount of time will the voters in N'' have enough money to purchase p''_1 and p''_2. Yet, at this time the voters in N' would, altogether, have more than $4time''$. So, if p'_3 increased its cost to be between $3 \cdot time'$ and $4 \cdot time'$ (which is smaller than $4 \cdot time''$), then p'_3 would be funded before p''_1 and p''_2, and there would be sufficient amount of budget left for this.

Case B″ On the other hand, if at least one of p''_1 and p''_2 is preferred to p''_3 by tie-breaking at time $time''$, then both p''_1 and p''_2 are funded at that time (because after one of them is selected due to tie-breaking, the voters in V'' no longer have enough money to purchase p''_3, but they do have enough for the other one among p''_1 and p''_2). Consequently, after p''_1 and p''_2 are purchased at time $time''$, only projects p'_3 and p''_3 are not selected yet and the voters have the following amounts of money:

1. Voter $v_{3'}$ has $time''$ amount of money and the remaining voters in V' have nonzero amounts of money (specifically, each of them has $time'' - time'$, because they paid for p'_1 and p'_2 at time $time'$).

2. Voter $v_{3''}$ has $time''$ amount of money and the remaining voters in V'' have no money.

Since voters in V' have, together, more money than those in V'', but voters from both groups (jointly) earn money at the same rate (as $|V'| = |V''|$), if p'_3 increased its cost to be slightly below that of p''_3, it will be funded before p'_3. Consequently, if $\mathbf{c}^*$ is an equilibrium then we must have $time' = time''$.

Finally, it suffices to show that the assumption $time' = time''$ also leads to no equilibrium. W.l.o.g., we assume that p'_3 precedes both p'_1 and p'_2 in the tie-breaking order, and that p''_3 precedes both p''_1 and p''_2 (if this were not the case, then the arguments given in Case A above would still show that $\mathbf{c}^*$ is not an equilibrium). However, now we can see that, e.g., it is beneficial for p'_3 to slightly increase its cost. This follows by the same reasoning as given in Case B$'$. Hence, we have reached a contradiction. Consequently, under Phragmén there is no Nash equilibrium in our game. □

B.7 Proof of Theorem 4.9

Proof. Let (P, V, B, d) be a MES-Cost PB game.

We provide an iterative algorithm that computes $\mathbf{c}$. In each iteration, the algorithm fixes selected values of $\mathbf{c}$, drops the corresponding projects from further consideration and deletes the voters that have no budget left. The algorithm finishes when there is no more projects to consider. Before laying out the details, let us recall that in MES-Cost, each voter gets the equal share of the budget and cannot spend on the projects more than their entitlement.

Our algorithm constructs the strategy $\mathbf{c}$ step by step maintaining the collection V' of voters to consider and the collection P' of projects to consider. The algorithm starts with setting $V' = V$, $P' = P$ and proceeds as follows:

1. Remove from P' all projects $p_i \in P'$ for which $|A(p_i) \cap V'| \cdot {}^{B}\!/_{|V|}$ is lower than $d(p_i)$ or equal to zero and let their strategy be $\mathbf{c}(p_i) = d(p_i)$.

2. For each project $p_j \in P'$, let $\alpha_j = {}^{1}\!/_{|A(p_j) \cap V'|}$ (due to Step 1 we avoid dividing by zero) and let p^* be the project $p_j \in P'$ with the minimum α_j-value (in case of a tie, select the first one in the tie-breaking order).

3. Set the strategy $\mathbf{c}(p^*)$ to $|A(p^*) \cap V'| \cdot {}^{B}\!/_{|V|}$, that is, such that p^* reports the cost equal to the total budget of its supporters in V'.

4. Remove all supporters of p^* from V' and remove p^* from P'.

5. Repeat Steps 1 to 4 until P' is empty.

Our algorithm is based on the following useful claim about the decisions made by MES-Cost.

Claim 2. *Let us fix some stage of MES-Cost in which every voter with a nonzero budget has the same value of budget and let V' be these voters. Then, for each α'-affordable project p' it holds that that $\alpha' = {}^{1}\!/_{|A(p') \cap V'|}$.*

The claim is implied by the definition of MES-Cost. Observe that if a project p is α-affordable, then the voters approving it have enough budget to buy it and α is the minimum such x that $|A(p') \cap V'| \cdot x \cdot cost(p) \leq cost(p)$. The sought α is then clearly ${}^{1}\!/_{|A(p') \cap V'|}$.

Let us denote by $(p^*_1, p^*_2, \ldots, p^*_k)$ the projects considered in the respective iterations 1 to k of Steps 1-4. In what follows we show that in each Nash equilibrium MES-Cost selects exactly projects $(p^*_1, p^*_2, \ldots, p^*_k)$ in the given order and that $\mathbf{c}$ is a Nash equilibrium. We apply induction over the stages of MES-Cost.

For the base case, we argue that p^*_1 has to be selected first by MES-Cost and has to play $\mathbf{c}(c^*_1)$ in every Nash equilibirum. Assume for contradiction that some other project p' is selected first instead

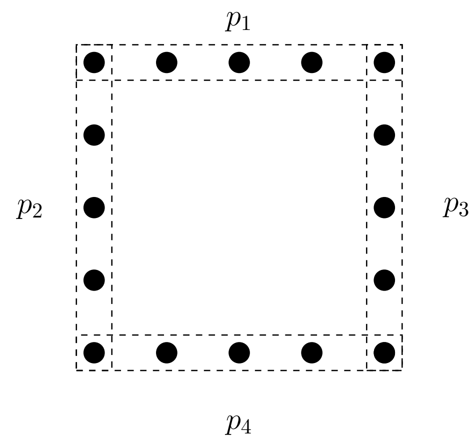

Figure 4: Approval sets of projects p_1, p_2, p_3, and p_4. Here, each vertex represents a single voter.

of p_1^* in a Nash equilibrium. Due to Claim 2 and since MES-Cost selects α-affordable projects starting from those with the smallest α, it follows that one of the three holds: (i) $1/|A(p')| < 1/|A(p_1^*)|$, (ii) $1/|A(p')| = 1/|A(p_1^*)|$ and p' is preferred to p_1^* by the tie-breaking, or (iii) p_1^* reports a cost greater than the budget of its supporters. Cases (i) and (ii) yield a clear contradiction to Step 2, which defines p_1^*. In Case (iii), reporting cost $\mathbf{c}'(p_1^*) = d(p_1^*) + \epsilon \leq B \cdot |A(p_1^*)|/|V|$ (such an ϵ always exists due to Step 1) is a profitable deviation for p_1^*—a contradiction. Knowing that p_1^* is selected first in every Nash equilibrium, we observe that if p_1^* reports a cost $\mathbf{c}'(p_1^*) < B \cdot |A(p_1^*)|/|V|$, then reporting exactly $\mathbf{c}(p_1^*) = B \cdot |A(p_1^*)|/|V|$ is always a profitable deviation, which confirms that in every Nash equilibrium p_1^*'s strategy is $\mathbf{c}(p_1^*)$ and it is selected first by MES-Cost.

We move on to the inductive step and thus consider stage i of MES-Cost in which project p_i^* is selected. Imporantly, as we have shown that player p_1^* has to report cost $\mathbf{c}(p_1^*)$ equal to the total budget of p_1^* supporters, then we can assume that at the i-th stage we have only voters who either spend all their budget or who did not spend their budget at all; we denote the latter group by V'. As a result, Claim 2 holds at the i-th stage, which we consider. Thanks to this, we apply an analogous exchange argument (pretending that voters outside of V' do not exist) to that for the base case to show that indeed p_i^* has to be selected at the i-th stage. Then, we directly repeat the argument regarding the optimal strategy for p_1^* applying it to p_i^*. Eventually, we have shown that in each Nash equilibrium for MES-Cost projects $(p_1^*, p_2^*, \ldots, p_k^*)$ are selected and report, respectively, costs $(\mathbf{c}(p_1^*), \mathbf{c}(p_2^*), \ldots, \mathbf{c}(p_k^*))$. □

B.8 Proof of Theorem 4.10

Proof. Let (P, V, B, d) be a MES-Apr PB game with plurality ballots. In MES-Apr, like in every MES rule, each voter receives $B/|V|$ money for the whole election process. As each voter approves only one project, each project p_i can request at most $M(p_i) = |A(p_i)| \cdot B/|V|$ from its supporters, and will report this cost as they would not spend any money on other projects. Thus, if $d(p_i) \leq M(p_i)$, project p_i reports $M(p_i)$ and gets selected with its maximum possible cost. Otherwise, if $d(p_i) > M(p_i)$, p_i cannot be selected with cost covering $d(p_i)$, so p_i reports $d(p_i)$ and is not selected. □

B.9 Proof of Proposition 4.11

Proof. We create four projects $p_1, \ldots, p_4$ and 16 voters $v_1, \ldots, v_{16}$. Project p_1 is approved by voters $v_1, \ldots, v_5$; project p_2 by voters $v_{13}, \ldots, v_{16}, v_1$; project p_3 by voters $v_5, \ldots, v_9$; and project p_4 by voters $v_9, \ldots, v_{13}$. We set the budget to be some positive integer B and delivery costs to be $d \equiv 0$. Please note that we do not specify tie-breaking as we prove nonexistence of MES-Apr-NE in any tie-breaking.

For a better visualization of the instance, please look at the Figure 4. In particular, one can see that the instance is symmetric (each project is approved by three voters supporting only it and by two voters, one shared with each neighbour).

Now we will show that there is no MES-Apr-NE for this instance. Suppose towards contradiction that some profile $\mathbf{c}$ is a MES-Apr-NE.

At the beginning, each voter receives $b = \frac{B}{|V|} = \frac{B}{16}$ money. Therefore, a) no project says cost exceeding $5b = \frac{5B}{16}$ (otherwise its supporters would never have enough money to buy it, and b) no project says less than $3b = \frac{3B}{16}$ (as each project has three supporters that contribute only to it, it is nonoptimal to say less than they have). Further, as each project has zero delivery costs, each project must be selected in MES-Apr-NE, otherwise an unselected project could have decreased its cost to 0 and be selected. For this reason, $\sum_{i=1}^{4} \mathbf{c}(p_i) \leq B$.

For the sake of brevity, for yet-unselected project p_i in iteration j, we denote $\alpha_j(p_i)$ as minimum value such that project p_i is $\alpha_j(p_i)$-affordable according to MES-Apr rule (or infinity, if such a value does not exist). By $\epsilon \in \mathbb{R}_+$ we mean some very small positive real number such that adding it does not change the given strict inequality sign.

Due to symmetry, w.l.o.g. we can assume that 1) project p_1 says the lowest cost (i.e., $\mathbf{c}(p_1) \leq \mathbf{c}(p_2), \mathbf{c}(p_3), \mathbf{c}(p_4)$) and wins ties if there is more than more project with this cost as well as 2) $\mathbf{c}(p_2) \leq \mathbf{c}(p_3)$ and p_2 wins ties with p_3. If it was not the case, we can shift or rotate projects and perform analogous reasoning.

Before we move on, let us exclude some cases in which $\mathbf{c}$ cannot be MES-Apr-NE.

We point out that if there is no project of the same cost as p_1, then p_1 could have increased its cost by some positive number ϵ while still being considered first and getting more, so $\mathbf{c}$ could not have been MES-Apr-NE in this case. Therefore, p_1 has the same cost either as 1) p_4, or as 2) p_2 (please note $\mathbf{c}(p_1) = \mathbf{c}(p_3)$ implies $\mathbf{c}(p_1) = \mathbf{c}(p_2)$ due to assumption that $\mathbf{c}(p_2) \leq \mathbf{c}(p_3)$).

With our assumptions, project p_1 is selected in the first iteration and each of its supporters pays $\frac{\mathbf{c}(p_1)}{5}$ for p_1. As $b - \frac{\mathbf{c}(p_i)}{5} \leq \frac{B}{16} - \frac{\frac{3B}{16}}{5} = \frac{\frac{2B}{16}}{5} < \frac{\frac{3B}{16}}{5} \leq \frac{\mathbf{c}(p_j)}{5}$ for any $p_i, p_j \in P$, voter v_1 has insufficient money to contribute the equal share $\frac{\mathbf{c}(p_2)}{5}$ to purchase p_2. Therefore, in the second iteration, $\alpha_2(p_4) = \frac{\mathbf{c}(p_4)}{5}$, $\alpha_2(p_2) = \frac{\mathbf{c}(p_2) - (b - \frac{\mathbf{c}(p_1)}{5})}{4}$, and $\alpha_2(p_3) = \frac{\mathbf{c}(p_3) - (b - \frac{\mathbf{c}(p_1)}{5})}{4} \geq \alpha_2(p_2)$. Please note that $\alpha_2(p_2) = \frac{\mathbf{c}(p_2) - (b - \frac{\mathbf{c}(p_1)}{5})}{4} \geq \frac{\mathbf{c}(p_1) - (\frac{B}{16} - \frac{\mathbf{c}(p_1)}{5})}{4} = \frac{6\mathbf{c}(p_1)}{20} - \frac{B}{64} = \frac{4\mathbf{c}(p_1)}{20} + (\frac{2\mathbf{c}(p_1)}{20} - \frac{B}{64}) \geq \frac{\mathbf{c}(p_1)}{5} + (\frac{2 \cdot \frac{3B}{16}}{20} - \frac{B}{64}) = \frac{\mathbf{c}(p_1)}{5} + \frac{6B - 5B}{16 \cdot 20} = \frac{\mathbf{c}(p_1)}{5} + \frac{B}{16 \cdot 20} > \frac{\mathbf{c}(p_1)}{5} = \alpha_1(p_1)$. Thus, as $\alpha_2(p_2) > \alpha_1(p_1)$, project p_4 would regret saying the same cost as project p_1. Indeed, selecting p_1 in the first iteration significantly increases $\alpha_2(p_2)$ and $\alpha_2(p_3)$, so if p_4 said $\mathbf{c}(p_4) = \mathbf{c}(p_1)$, then it would benefit from slightly increasing the cost by ϵ (it would still be before p_2, but it would earn more). For this reason, it cannot be the case that $\mathbf{c}(p_1) = \mathbf{c}(p_4)$, so due to the former reasoning we know that $\mathbf{c}(p_1) = \mathbf{c}(p_2)$ and $\mathbf{c}(p_1) < \mathbf{c}(p_4)$.

So we know that: $\mathbf{c}(p_1) = \mathbf{c}(p_2)$, $\mathbf{c}(p_1) < \mathbf{c}(p_4)$, and $\mathbf{c}(p_2) \leq \mathbf{c}(p_3)$. We have two cases to consider:

- Project p_4 is selected in the second iteration. It means that $\alpha_2(p_4) \leq \alpha_2(p_2)$. We observe that it must be $\alpha_2(p_4) = \alpha_2(p_2)$, otherwise it would be beneficial for p_4 to slightly increase its cost by ϵ in such a way that p_4 is still considered before p_2 and gains more.

  In this case, as p_2 and p_3 are approved by disjoint sets of voters and both p_1 and p_4 took the same amount of money from the same number of their supporters, both p_2 and p_3 should request for the same amount of money, that is, all money that their supporters have. Thus $\mathbf{c}(p_3) = \mathbf{c}(c_2) = 3 \cdot b + (b - \frac{\mathbf{c}(p_1)}{5}) + (b - \frac{\mathbf{c}(p_4)}{5}) = 5b - \frac{\mathbf{c}(p_1)}{5} - \frac{\mathbf{c}(p_4)}{5}$ and $\mathbf{c}(p_4) = 5 \cdot (5b - \frac{\mathbf{c}(p_1)}{5} - \mathbf{c}(p_2)) = 25b - 5\mathbf{c}(p_2) - \mathbf{c}(p_1)$.

  However, we know that $\mathbf{c}(p_1) = \mathbf{c}(p_2)$, so $\mathbf{c}(p_1) = \mathbf{c}(p_2) = \mathbf{c}(p_3) = 5b - \frac{\mathbf{c}(p_1)}{5} - \frac{\mathbf{c}(p_4)}{5}$, which implies that $\mathbf{c}(p_4) = 25b - 5\mathbf{c}(p_2) - \mathbf{c}(p_1) = 25b - 6\mathbf{c}(p_1)$. Therefore, $\alpha_2(p_4) = \alpha_2(p_2)$ means that $\frac{\mathbf{c}(p_4)}{5} = \frac{\mathbf{c}(p_2) - (b - \frac{\mathbf{c}(p_1)}{5})}{4} = \frac{6\mathbf{c}(p_1)}{20} - \frac{b}{4}$, which is equivalent to $\frac{25b - 6\mathbf{c}(p_1)}{5} = \frac{6\mathbf{c}(p_1) - 5b}{20} \Rightarrow 100b - 24\mathbf{c}(p_1) = 6\mathbf{c}(p_1) - 5b \Rightarrow 30\mathbf{c}(p_1) = 105b \Rightarrow \mathbf{c}(p_1) = \frac{7b}{2}$. Then we have $\mathbf{c}(p_1) = \mathbf{c}(p_2) = \mathbf{c}(p_3) = \frac{7b}{2}$ and $\mathbf{c}(p_4) = 25b - 6\mathbf{c}(p_1) = 25b - 21b = 4b$.

Now imagine that project p_1 changes its cost from $\mathbf{c}(p_1) = \frac{7b}{2}$ to $\mathbf{c}'(p_1) = \frac{36b}{10}$. Naturally, projects p_2 and p_3 are selected in first two iterations so voters $v_1, \ldots, v_5$ supporting p_1 are left with $b - \frac{\mathbf{c}(c_2)}{5}, b, b, b, b - \frac{\mathbf{c}(p_3)}{5}$ money respectively and they have $b - \frac{\mathbf{c}(c_2)}{5} + b + b + b + b - \frac{\mathbf{c}(p_3)}{5} = 5b - 2 \cdot \frac{7b}{10} = \frac{36b}{10}$ money in total which they can only spend on project p_1 as they disapprove p_4. So p_1 gets selected with $\mathbf{c}'(p_1) = \frac{36b}{10} > \frac{7b}{2} = \mathbf{c}(p_1)$ which indicates that $\mathbf{c}$ could not be MES-Apr-NE.

- Project p_2 is selected in the second iteration. It means that $\alpha_2(p_2) \leq \alpha_2(p_4)$. We observe that it must be $\alpha_2(p_2) = \alpha_2(p_4)$, otherwise it would be beneficial for p_2 to slightly increase its cost in such a way that p_2 is still considered before p_4 and gains more (please note that even if p_3 would jump before p_2, it takes money from the disjoint set of voters from p_2's supporters, so it does not disprove this argument).

  By performing analogous calculations to the previous case and using $\mathbf{c}(p_1) = \mathbf{c}(p_2)$ we obtain that $\alpha_2(p_2) = \frac{\mathbf{c}(p_2) - (b - \frac{\mathbf{c}(p_1)}{5})}{4} = \frac{6\mathbf{c}(p_1)}{20} - \frac{b}{4}$, so $\alpha_2(p_4) = \alpha_2(p_2) \Rightarrow \frac{\mathbf{c}(p_4)}{5} = \frac{6\mathbf{c}(p_1)}{20} - \frac{b}{4} \Rightarrow \mathbf{c}(p_4) = \frac{6\mathbf{c}(p_1) - 5b}{4}$.

  After p_1 and p_2 are selected, voter v_1 has 0 money left, voters $v_2, \ldots, v_5$ have $b - \frac{\mathbf{c}(p_1)}{5}$ money left, voters $v_{13}, \ldots, v_{16}$ have $b - \alpha_2(p_2) = b - (\frac{6\mathbf{c}(p_1)}{20} - \frac{b}{4}) = \frac{25b - 6\mathbf{c}(p_1)}{20}$ money left, and voters $v_6, \ldots, v_{12}$ have b money left.

  We need to consider two cases:

  - p_3 is selected before p_4. Then $\mathbf{c}(p_3) = \mathbf{c}(p_2)$, otherwise (if $\mathbf{c}(p_2) < \mathbf{c}(p_3)$) p_2 would increase its cost by ϵ in such a way that $\mathbf{c}(p_2) + \epsilon < \mathbf{c}(p_3)$ and still be selected in the second iteration, but with more money. Because $\mathbf{c}(p_3) = \mathbf{c}(p_2) = \mathbf{c}(p_1)$ and the supporters of p_2 and p_3 are disjoint and symmetric, after selecting p_3, voter v_5 be left with no money whereas voters $v_6, \ldots, v_9$ with $b - \alpha_3(p_3) = b - \alpha_2(p_2) = \frac{25b - 6\mathbf{c}(p_1)}{20}$. Further, in the fourth iteration, the supporters of p_4 have together $3b + 2 \cdot \frac{25b - 6\mathbf{c}(p_1)}{20} = \frac{30b + 25b - 6\mathbf{c}(p_1)}{10} = \frac{55b - 6\mathbf{c}(p_1)}{10}$, so p_4 asks for the money they still have in total.

    Therefore, $\mathbf{c}(p_4) = \frac{6\mathbf{c}(p_1) - 5b}{4} = \frac{55b - 6\mathbf{c}(p_1)}{10} \Rightarrow 30\mathbf{c}(p_1) - 25b = 110b - 12\mathbf{c}(p_1) \Rightarrow 42\mathbf{c}(p_1) = 135b \Rightarrow \mathbf{c}(p_1) = \frac{135b}{42}$. Thus $\mathbf{c}(p_4) = \frac{6\mathbf{c}(p_1) - 5b}{4} = \frac{6 \cdot \frac{135b}{42} - 5b}{4} = \frac{135b - 35b}{4 \cdot 7} = \frac{100b}{28} = \frac{150b}{42}$.

    Imagine what happens if p_1 increases its cost from $\mathbf{c}(p_1) = \frac{135b}{42}$ to $\mathbf{c}'(p_1) = \frac{156b}{42}$. Then p_2 gets selected first with cost $\mathbf{c}(p_2) = \frac{135b}{42}$ and each of its supporters pays $\frac{\mathbf{c}(p_2)}{5} = \frac{\frac{135b}{42}}{5} = \frac{27b}{42}$. Next, each of p_3's supporters will pay $\frac{27b}{42}$ to purchase p_3 (voters supporting p_2 and p_3 are disjoint). After that, regardless whether p_4 is selected before p_1 or not, the supporters of p_1 have $3b + 2 \cdot (b - \frac{27b}{42}) = \frac{3 \cdot 42 + 2 \cdot 15b}{42} = \frac{156b}{42}$, so the supporters of p_1 have enough money to purchase p_1. We showed that p_1 could improve its utility by changing its cost, so the proposed $\mathbf{c}$ is not MES-Apr-NE in this case.

  - p_4 is selected before p_3. It means that $\alpha_3(p_4) \leq \alpha_3(p_3)$. As we know how much money each voter has in the third iteration, $\alpha_3(p_4) = \frac{\mathbf{c}(p_4) - \frac{25b - 6\mathbf{c}(p_1)}{20}}{4} = \frac{20\mathbf{c}(p_4) - 25b + 6\mathbf{c}(p_1)}{80}$ and $\alpha_3(p_3) = \frac{\mathbf{c}(p_3) - \frac{\mathbf{c}(p_1)}{5}}{4} = \frac{20\mathbf{c}(p_3) - 4\mathbf{c}(p_1)}{80}$. After inserting the exact values we obtain that $\alpha_3(p_4) \leq \alpha_3(p_3) \Rightarrow \frac{20\mathbf{c}(p_4) - 25b + 6\mathbf{c}(p_1)}{80} \leq \frac{20\mathbf{c}(p_3) - 4\mathbf{c}(p_1)}{80} \Rightarrow 20\mathbf{c}(p_3) \geq 20\mathbf{c}(p_4) - 25b + 10\mathbf{c}(p_1) \Rightarrow \mathbf{c}(p_3) \geq \mathbf{c}(p_4) + \frac{\mathbf{c}(p_1)}{2} - \frac{5b}{4}$. However, $\mathbf{c}(p_3) \geq \mathbf{c}(p_4) + \frac{\mathbf{c}(p_1)}{2} - \frac{5b}{4} > \mathbf{c}(p_4) + \frac{3b}{2} - \frac{5b}{4} = \mathbf{c}(p_4) + \frac{b}{4} > \mathbf{c}(p_1) + \frac{b}{4} = \mathbf{c}(p_2) + \frac{b}{4}$, so $\mathbf{c}(p_3)$ is significantly greater than $\mathbf{c}(p_2)$. In other words, even when voter v_9 pays a greater share to buy p_4 and is left with less money than $\frac{\mathbf{c}(p_4)}{5}$, there is still enough money to purchase p_3. Therefore, p_2 can increase its cost from $\mathbf{c}(p_2) = \mathbf{c}(p_1)$ to $\mathbf{c}(p_3) - \epsilon$ and lose in the second iteration with p_4, but still be funded before p_3 and obtain more money. For this reason, $\mathbf{c}$ is not MES-Apr-NE.

We showed that in no case MES-Apr-NE exists for this instance. The remaining tie-breaking order are completely analogous due to the instance's symmetry.

It means that for the given instance there is no tie-breaking for which MES-Apr-NE exists. $\square$

B.10 Proof of Theorem 4.12

Proof. We first consider the case of zero delivery costs, as it is simpler. Let (P, N, B, d) be a MES-Apr PB game with party-list ballots and zero delivery costs.

At the beginning, each voter receives $B/|V|$ money. In a party-list profile, as every two voters have either the same preferences or the disjoint ones, $A(p_i) = A(party(p_i))$. The total money the supporters of $party(p_i)$ have is $M(party(p_i)) = |A(p_i)| \cdot B/|V| = \frac{|A(p_i)| \cdot B}{|V|}$. Therefore, the projects in $party(p_i)$ need to somehow distribute this money between them in such a way that no project would complain and change its cost to obtain better outcome.

Since we have cardinal utilities, the α-value of project p_i is initially $\alpha_1(p_i) = \frac{\mathbf{c}(p_i)}{|A(party(p_i))|}$. For this reason, MES-Apr will start from the cheapest project in the party to the most expensive one (following tie-breaking order in case of a tie) and equally take the budget from the party supporters to fund the next project unless the cost exceeds the money they still have.

This means that each project should ask for the equal part of the money of party supporters, that it, $\frac{M(party(p_i))}{|party(p_i)|}$. The project asking for more money would be moved to the end of the order when its supporters have insufficient money to fund it, so no project would increase its cost. Asking for less money is pointless if it is already funded.

For this reason, for the profile $\mathbf{c}$ where each project says $\mathbf{c}(p_i) = \frac{M(party(p_i))}{|A(party(p_i)|}$, we obtain MES-Apr-NE.

Next we move to the case with arbitrary delivery costs. Let (P, V, B, d) be a MES-Apr PB game with party-list ballots. Recall that at the beginning each voter receives $B/|V|$ money. Then, the total money the supporters of $party(p_i)$ have is $M(party(p_i)) = |A(p_i)| \cdot B/|V| = \frac{|A(p_i)| \cdot B}{|V|}$.

Let $\succ$ be the A/D tie-breaking order from the theorem statement. Note that for the case of party-list ballots $\succ$ orders the projects within the same party nondescendingly by the delivery costs. That is, within the same party cheaper projects preferred to the more expensive ones. In case of two projects within the same party with equal delivery costs, we assume without loss of generality that the project with a lower index is preferred (we can always relabel projects to achieve this condition). Notably, $\succ$ does not impose any specific order with respect to the delivery costs over any two projects of two different parties.

Since in MES-Apr we consider cardinal utilities, the α-value of project p_i is at the beginning: $\alpha_1(p_i) = \frac{\mathbf{c}(p_i)}{|A(party(p_i))|}$. For this reason, MES-Apr will start from the cheapest project in the party to the most expensive one (following the specified tie-breaking order in case of a tie) and equally take the budget from the party supporters to fund the next project unless the cost exceeds the money they still have.

Now we have to consider two cases:

- The last in the tie-breaking order project $p_j \in party(p_i)$ has delivery cost $d(p_j)$ not exceeding $\frac{M(party(p_i))}{|party(p_i)|}$. Note that this case is similar to our case with zero delivery costs that we started this proof with, i.e., each project submits $\frac{M(party(p_i))}{|party(p_i)|}$, gets funded, and no project benefits from changing its cost.
- The last in the tie-breaking order project $p_j \in party(p_i)$ has delivery cost $d(p_j)$ exceeding $\frac{M(party(p_i))}{|party(p_i)|}$. In such a case, if every project submitted cost $d(p_j)$, then p_j would not be selected as it is last in the tie-breaking order and cannot profitably decrease its cost. For this reason, we set the cost of p_j to be $d(p_j)$, remove p_j from consideration, and repeat this procedure as long as the product of the number of yet-remained projects and the delivery

cost of the last in the tie-breaking project exceeds $M(party(p_i))$. Suppose now that the procedure stopped and project p_k was the last removed one. Then all y yet-remaining projects would say cost $min({}^{M(party(p_i))}/_{y}, d(p_k))$ and be given this funding. Saying more than $d(p_k)$ would result in being overtaken by project p_k and thrown out of the outcome due to insufficient budget. Saying more than ${}^{M(party(p_i))}/_{y}$ would result in getting the last among remaining projects and asking for more money than left in the budget at that timestamp.

The combination of $\mathbf{c}$ profiles calculated in the above way for all parties is a MES-Apr-NE for the given instance. $\square$

Example B.2. *Take a PB game G with one voter approving projects p_1, p_2, and p_3. Let $B = 6$, $d(p_1) = 3$, $d(p_2) = 0$, and $d(p_3) = 0$, with $p_1 \succ p_2 \succ p_3$. There is no MES-Apr-NE in G (the full proof is in the appendix and we give a sketch): Suppose that $\mathbf{c}$ is an NE. Then $\mathbf{c}(p_1) \geq 3$ (otherwise, as it is selected, p_1 improves by reporting 3) and $\mathbf{c}(p_2), \mathbf{c}(p_3) < \mathbf{c}(p_1) = 3 = {}^{B}/_{2}$ (otherwise, the more expensive one is not funded). Since $\mathbf{c}(p_2) + \mathbf{c}(p_3) < B$, p_2 may report ${}^{(\mathbf{c}(c_1)+\mathbf{c}(c_2))}/_{2}$ and still be selected, so $\mathbf{c}$ is not an NE.*

Suppose towards contradiction that $\mathbf{c}$ is such an equilibrium. W.l.o.g.,let $\mathbf{c}(p_1) \geq d(p_1) = 3$. Otherwise, if p_1 would be selected under these costs (and, hence, received a negative payoff), then it would prefer to report $d(p_1)$ and receive utility 0. If it was not funded under $\mathbf{c}$, then $(\mathbf{c}_{-i}, d(p_1))$ also would be an equilibrium. Next, we observe that $\mathbf{c}(p_1) = 3$. Indeed, if p_1 reported a value greater than 3, then one of the projects could obtain a better payoff: If p_2 or p_3 reported a value greater or equal to $\mathbf{c}(p_1)$, then it would benefit by reporting a smaller one; if p_2 and p_3 reported values that sum up to more than B, then at least one of them would benefit by reporting a smaller cost, and if they reported values that sum up to at most B then either (a) one of them would benefit by reporting a larger cost, or (b) if they both reported 3, p_1 would benefit by reporting 3. Hence, we have that $\mathbf{c}(p_1) = 3 = {}^{B}/_{2}$. Then, both $\mathbf{c}(p_2) > 0$ and $\mathbf{c}(p_3) > 0$ (the project reporting 0 would benefit by reporting a larger number). Further, we have that $\mathbf{c}(p_2) < \mathbf{c}(p_1)$ and $\mathbf{c}(p_3) < \mathbf{c}(p_1)$. Indeed, if either p_2 or p_3 reported value greater or equal to $\mathbf{c}(p_1)$ then it would not be selected and, hence, would benefit by reporting a smaller cost. So, $\mathbf{c}(p_2) + \mathbf{c}(p_3) < B$. But then either of them would benefit by reporting a higher cost (so that the sum of their costs would not exceed B). Thus $\mathbf{c}$ is not a MES-Apr-NE.

B.11 Proof of Theorem 4.13

Proof. Our example is in fact the example from Proposition 4.8 narrowed to prim voters and projects.

We have three voters v_1, v_2, and v_3 as well as three projects p_1, p_2, and p_3. Projects p_1 and p_2 are approved by, respectively, voters v_1 and v_2, while project p_3 is approved by all three voters. We set the budget to be some positive integer denoted as B and the delivery costs of these projects to be $d \equiv 0$. We set the tie-breaking between projects to be $p_1 \succ p_2 \succ p_3$. For the sake of brevity, for yet-unselected project p_i in iteration j, we denote $\alpha_j(p_i)$ as minimum value such that project p_i is $\alpha_j(p_i)$-affordable according to MES-Apr rule (or infinity, if such a value does not exist).

Let $(\mathbf{c}(p_1) = \frac{7B}{36}, \mathbf{c}(p_2) = \frac{8B}{36}, \mathbf{c}(p_3) = \frac{21B}{36})$ be the costs the projects say. We will show that it is MES-Apr-NE.

At the beginning, each of three voters receives $\frac{B}{3} = \frac{12B}{36}$ money.

In the first iteration, $\alpha_1(p_1) = \frac{7B/36}{1} = \frac{7B}{36}, \alpha_1(p_2) = \frac{8B/36}{1} = \frac{8B}{36}, \alpha_1(p_3) = \frac{21B/36}{3} = \frac{7B}{36}$, so p_1 and p_3 are tied, while p_2 has greater α-value and is skipped for now. Thus, due to tie-breaking, we select p_1 and voter v_1 pays $\frac{7B}{36}$ for p_1.

In the second iteration, v_1 has $\frac{5B}{36}$ money while both v_2 and v_3 have $\frac{12B}{36}$. Thus, since the budget of v_2 did not change, $\alpha_2(p_2) = \alpha_1(p_2) = \frac{8B}{36}$. In order to purchase p_3, voter v_1 would need to spend all left money whereas v_2 and v_3 would need to pay more to make up v_1's insufficient money. More precisely, $\alpha_2(p_3) = \frac{21B/36 - 5B/36}{2} = \frac{16B/36}{2} = \frac{8B}{36}$. So projects p_2 and p_3 are tied and we select p_2 due to tie-breaking (by taking money from v_2).

Finally, in the third iteration, we will select p_3 as its supporters have $\frac{5B}{36} + \frac{4B}{36} + \frac{12B}{36} = \frac{21B}{36} = \mathbf{c}(p_3)$, all voters will use use all their money for it.

We showed that each project is selected with the proposed costs so no project benefits from lowering its costs. Next, p_3 will not increase its cost as its supporters would not have enough money to buy it in the last iteration. Further, p_2 will not increase its cost as it would lose with p_3 in the second iteration and its only supporter v_2 would pay $\frac{8B}{36}$ for p_3, leaving insufficient $\frac{4B}{36}$ for p_2. Analogously, p_1 will not increase its cost as it would lose with p_3 in the first iteration and its only supporter v_1 would pay $\frac{7B}{36}$ for p_3, leaving insufficient $\frac{5B}{36}$ for p_1.

For this reason, $\mathbf{c} = (\mathbf{c}(p_1) = \frac{7B}{36}, \mathbf{c}(p_2) = \frac{8B}{36}, \mathbf{c}(p_3) = \frac{21B}{36})$ is a MES-Apr-NE for the given instance.

One can show in an analogous way that $\mathbf{c}' = (\mathbf{c}'(p_1) = \frac{8B}{36}, \mathbf{c}'(p_2) = \frac{7B}{36}, \mathbf{c}'(p_3) = \frac{21B}{36})$ is also MES-Apr-NE for the given instance. □

B.12 Proof of Theorem 6.1

Let us begin by introducing additional notation. In a *cost-interval PB game* $G^+ = (P, V, B)$ we associate each voter v with an *approval function* $A : V \times P \to 2^{[0,B]}$, with $A(v, p)$ specifying how costly can a project p be for v to approve it. For simplicity, we assume that $A(v, p)$ is a possibly empty interval. We say that a voter v *potentially supports* a project p if $A(v, p) \neq \emptyset$. Then, for a strategy profile $\mathbf{c}$, the corresponding PB election $E(\mathbf{c}) = (P, V, B, \mathbf{c})$ is such that each voter v approves a project $p \in P$ if $cost(p) \in A_v(p)$. Then, the payoff function and Nash equilibria are defined for cost-interval PB games analogously to PB games. We now showcase examples witnessing the result for particular voting rules.

MES-Cost. Consider the following cost-interval PB game G. Take the set of projects $P = \{p_1, p_2\}$, as well as the set of voters $V = \{v_1, v_2, v_3, v_4\}$. Further, we let $A(v_1, p_1) = A(v_2, p_1) = [8, 12]$, $A(v_3, p_1) = A(v_4, p_1) = [0, 4]$, $A(v_1, p_2) = [4, 8]$, and $A(v_2, p_2) = A(v_3, p_2) = A(v_4, p_2) = [0, 3]$. Finally, let $B = 16$.

Suppose that there exists a strategy profile $\mathbf{c}$ that is a MES-Cost-NE. Let us then consider the following, exhaustive cases:

1. $\mathbf{c}(p_1) \in (4, 8) \cup (12, 16]$. Then, p_1 is not funded, while for $\mathbf{c}(p_1) = 1$, p_1 is founded regardless of $\mathbf{c}(p_2)$.
2. $\mathbf{c}(p_1) \in (8, 12]$ Then, p_1 is not funded, and hence could benefit from lowering its cost.
3. $\mathbf{c}(p_1) = 8$. Then, we note that as $\mathbf{c}$ is an NE, $\mathbf{c}(p_2) = 3$, as then p_2 is funded but would not be selected for $(8, \mathbf{c}'(p_2))$ with $\mathbf{c}'(p_2) > 3$. But then, p_1 is not funded under $\mathbf{c}$, while it would benefit from submitting $\mathbf{c}'(p_1) = 4$.
4. $\mathbf{c}(p_1) \in [0, 4]$. Then, $\mathbf{c}(p_2) = 4$, as then p_2 is funded under $\mathbf{c}$ but would not be under $(\mathbf{c}(p_1), \mathbf{c}'(p_2))$, for any $\mathbf{c}'(p_2) > 4$. However, in that case p_1 would benefit from raising their cost to 8.

It follows that there is no MES-Cost-NE in G.

BasicAV. Consider the following cost-interval PB game G. Take the set of projects $P = \{p_1, p_2\}$, where $p_1 \succ p_2$, as well as the set of voters $V = \{v_1, v_2, v_3, v_4, v_5\}$. Further, we let $A(v_1, p_1) = [5, 9]$, $A(v_2, p_1) = [3, 7]$, $A(v_3, p_2) = [4, 8]$, $A(v_4, p_2) = [5, 7]$, and $A(v_5, p_2) = [1, 6]$ so this is a plurality profile. Finally, let $B = 10$.

Suppose that there exists a strategy profile $\mathbf{c}$ that is a BasicAV-NE. Let us then consider the following, exhaustive cases:

1. $\mathbf{c}(p_1) \in [0, 3) \cup (9, 10]$. Then, p_1 is not funded but would benefit from submitting $\mathbf{c}'(p_2) = 3$, as then p_1 is always selected.
2. $\mathbf{c}(p_1) \in [3, 4) \cup (7, 9]$. Then, we note that as $\mathbf{c}$ is an NE, $\mathbf{c}(p_2) = 7$, as then p_2 is funded but would not be selected for $(\mathbf{c}(p_1), \mathbf{c}'(p_2))$ with $\mathbf{c}'(p_2) > 7$. But then, p_1 is not funded under $\mathbf{c}$, while it would benefit from submitting $\mathbf{c}'(p_1) = 7$.

3. $\mathbf{c}(p_1) \in [4, 7]$. Then, $\mathbf{c}(p_2) = 6$, as then p_2 is funded under $\mathbf{c}$ but would not be under $(\mathbf{c}(p_1), \mathbf{c}'(p_2))$, for any $\mathbf{c}'(p_2) > 6$. However, in that case p_1 would benefit from reducing their cost to 4.

It follows that there is no BasicAV-NE in G.

AV/cost. Consider the following cost-interval PB game G. Take the set of projects $P = \{p_1, p_2\}$, where $p_2 \succ p_1$, as well as the set of voters $V = \{v_1, v_2, v_3, v_4\}$. Further, we let $A(v_1, p_1) = [4, 9]$, $A(v_2, p_1) = A(v_3, p_1) = [8, 9]$, and $A(v_4, p_2) = [2, 6]$, so this is a plurality profile. Finally, let $B = 10$.

Suppose that there exists a strategy profile $\mathbf{c}$ that is a AV/Cost. Let us then consider the following, exhaustive cases:

1. $\mathbf{c}(p_1) \in [0, 4) \cup (9, 10]$. Then, p_1 is not funded but would benefit from submitting $\mathbf{c}'(p_2) = 5$, as then p_1 is always selected.
2. $\mathbf{c}(p_1) \in [8, 9]$. Then, we note that as $\mathbf{c}$ is an NE, $\mathbf{c}(p_2) = {}^{\mathbf{c}(p_1)}/_3$, as then p_2 is funded but would not be selected for $(\mathbf{c}(p_1), \mathbf{c}'(p_2))$ with $\mathbf{c}'(p_2) > {}^{\mathbf{c}(p_1)}/_3$. But then, p_1 is not funded under $\mathbf{c}$, while it would benefit from submitting $\mathbf{c}(p_1) \in [4, 3 \cdot \mathbf{c}(p_2))$.
3. $\mathbf{c}(p_1) \in [5, 8]$. Then, $\mathbf{c}(p_2) = min(\mathbf{c}(p_2), 6) \in [5, 6]$, as then p_2 is funded under $\mathbf{c}$ but would not be under $(\mathbf{c}(p_1), \mathbf{c}'(p_2))$, for any $\mathbf{c}'(p_2) > min(\mathbf{c}(p_2), 6)$. However, in that case p_1 would benefit from raising their cost to 9.
4. $\mathbf{c}(p_1) \in [4, 5)$. Then, $\mathbf{c}(p_2) = min(10 - \mathbf{c}(p_2), 6) \in [5, 6]$, as then p_2 is funded under $\mathbf{c}$ but would not be under $(\mathbf{c}(p_1), \mathbf{c}'(p_2))$, for any $\mathbf{c}'(p_2) > min(10 - \mathbf{c}(p_2), 6)$. However, p_1 would benefit from raising their cost to 9.

It follows that there is no AV/Cost in G.

Phragmén. The claim follows by the example for AV/Cost, as Phragménand AV/Cost are equivalent for plurality elections.

MES-Apr. Consider the following cost-interval PB game G. Take the set of projects $P = \{p_1, p_2\}$, as well as the set of voters $V = \{v_1, v_2, v_3\}$. Also, let $A(v_1, p_1) = A(v_1, p_2) = [12, 18]$. Then, let $A(v_2, p_1) = A(v_2, p_2) = [6, 12]$ and the approval function of v_3 be the same as the approval function of v_2. Finally, we let $B = 18$ and assume zero delivery costs. It's clear that each voter received ${}^{B}/_3 = 6$ money units at the beginning of the algorithm regardless of the costs of the projects.

Suppose that there is a MES-Apr-NE $\mathbf{c} = (c_1, c_2)$ for G. First, we can see that each of the projects is funded with the cost set to 6 regardless of the cost submitted by the other proposer. So, both projects are funded under (c_1, c_2). Hence, we notice that $c_1 \in [6, 12]$ and $c_2 \in [6, 12]$, as otherwise, either one of the projects would submit cost higher than 12 and be supported by only voter (whose budget is 6), or submit a cost lower than 6 and not be supported by anyone. So, in both scenarios one of the project would not be funded. Let us then consider the following exhaustive cases:

1. $c_1 = 12$ or $c_2 = 12$. Without loss of generality, let $c_1 = 12$. Then, we see that $c_2 = 8$ or $c_2 = 12$. Otherwise, if $c_2 \in (8, 12)$, then p_2 would not be funded and if $c_2 < 8$, then p_2 would be funded under $(12, c_2 + \epsilon)$, for some $\epsilon > 0$. But then, one of the projects is not funded under (c_1, c_2), as $c_1 + c_2 > B$. Hence, according to previous observations, (c_1, c_2) is not a MES-Apr-NE.
2. $c_1 \in [6, 12)$ and $c_1 \in [6, 12)$. Observe that for this cost range, only voters v_2 and v_3 may pay for these projects, while having 12 money units in total. Then, we observe that $c_1 + c_2 \leq 12$, as otherwise one of the projects would not be funded. Moreover, $c_1 = c_2$, as otherwise the proposer p_i submitting the lower cost would benefit from increasing c_i. Hence, in this case $c_1 = c_2 = 6$. But then p_1 would also be funded under $(12, 6)$, and hence (c_1, c_2) is not an MES-Apr-NE.

It follows that there is no MES-Apr-NE in G.

Table 3: Additional PB instances that we analyze in the experiments. Vote Len. means the average number of approvals in a ballot. Rule indicates the PB rule that was used in the original election.

Instance	$\lvert P\rvert$	$\lvert V\rvert$	Budget	Vote Len.	Rule
Bemowo	83	5181	4854279	10.8	BasicAV
Bielany	98	4957	5258802	9.8	BasicAV
Wilanow	35	2359	1516962	9.59	BasicAV
Wlochy	43	2221	1719224	9.51	BasicAV

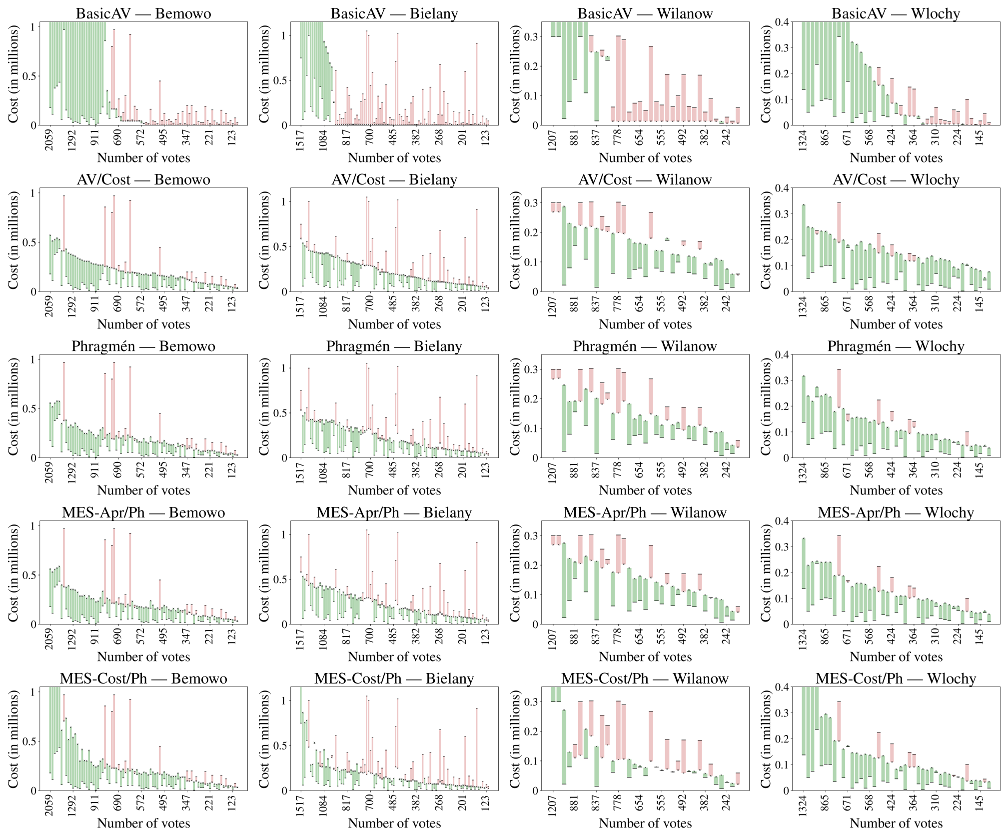

Figure 5: Winning (green bars) and losing (red bars) margins in real-life PB. Projects ordered by decreasing approval scores (on the x-axis) are represented by bars. Tacks and crosses show, respectively, the original and best response costs.

C Additional Experimental Results

Here, we present experimental results for four additional PB instances held in 2022 in different districts of Warsaw, Poland, i.e., in Bemowo, Bielany, Wilanow, and Wlochy. Details regarding these datasets are presented in Table 3. In Figure 5 we present the winning and losing margins in additional real-life PB instances, and in Figure 6 we present the strategy profiles after $10'000$ iterations of our dynamics.

Moreover, in Table 4 we present comparison of the original winning and losing margins and those after 10’000 iterations. We observe that in most cases the size of the margins drastically decreased.

D Dynamics With 80% Delivery Costs

In Figure 7 we show the results of the dynamics from Section 5.2 for the case where each project has delivery cost equal to the 80% of the cost it had in the original instance. We see that aside from

	Rule	Original Margins		10000 it. Margins	
		Winning	Losing	Winning	Losing
Bemowo	BasicAV	1830 ± 1283	122 ± 182	0	24 ± 37
	AV/Cost	150 ± 104	225 ± 257	1 ± 3	0.2 ± 0.1
	Phragmén	141 ± 91	220 ± 261	3 ± 3	0.9 ± 0.7
	MES-Apr/Ph	150 ± 95	223 ± 264	3 ± 4	2 ± 2
	MES-Cost/Ph	250 ± 323	205 ± 244	1 ± 2	1 ± 2
Bielany	BasicAV	1447 ± 1355	218 ± 205	0	2 ± 2
	AV/Cost	157 ± 128	171 ± 150	0.9 ± 2	0.2 ± 0.3
	Phragmén	146 ± 121	176 ± 154	4 ± 3	0.9 ± 0.9
	MES-Apr/Ph	148 ± 121	176 ± 151	3 ± 4	1 ± 2
	MES-Cost/Ph	172 ± 193	180 ± 154	3 ± 6	3 ± 4
Wesola	BasicAV	265 ± 249	64 ± 51	0	0
	AV/Cost	86 ± 27	55 ± 31	0.3 ± 0.4	0.2 ± 0.1
	Phragmén	69 ± 31	51 ± 37	2 ± 1	0.9 ± 1
	MES-Apr/Ph	79 ± 33	43 ± 37	2 ± 1	0.7 ± 0.7
	MES-Cost/Ph	70 ± 56	65 ± 45	0.1 ± 0.1	0.4 ± 0.5
Wilanow	BasicAV	536 ± 399	87 ± 80	0	0
	AV/Cost	87 ± 59	51 ± 34	2 ± 2	0.1 ± 0.0
	Phragmén	77 ± 52	66 ± 41	2 ± 2	0.3 ± 0.2
	MES-Apr/Ph	89 ± 56	58 ± 37	2 ± 1	0.7 ± 0.8
	MES-Cost/Ph	79 ± 120	95 ± 64	1 ± 2	2 ± 4
Wlochy	BasicAV	532 ± 524	46 ± 35	0	0
	AV/Cost	107 ± 49	56 ± 48	0.8 ± 1.0	0.1 ± 0.1
	Phragmén	83 ± 52	62 ± 42	3 ± 3	2 ± 1
	MES-Apr/Ph	85 ± 53	64 ± 45	2 ± 2	1 ± 1
	MES-Cost/Ph	138 ± 186	56 ± 46	0.2 ± 0.2	0.4 ± 0.5
Kleine Wereld	BasicAV	117 ± 78	14 ± 11	0	1.0 ± 0.0
	AV/Cost	12 ± 8	12 ± 12	0.1 ± 0.1	0.1 ± 0.0
	Phragmén	10 ± 8	10 ± 11	0.5 ± 0.6	0.2 ± 0.2
	MES-Apr/Ph	11 ± 7	11 ± 11	0.4 ± 0.6	0.2 ± 0.2
	MES-Cost/Ph	26 ± 34	11 ± 11	2 ± 1	0.1 ± 0.0

Table 4: Comparison of winning and losing margins in the original elections and after 10000 iterations of the game. (All the values are in 1000 PLN\EUR). In each entry the first value denotes the average and the second value (i.e., the one after $\pm$ sign) denotes the standard deviation.

fairly expensive projects, whose cost could not drop below the delievery cost, the overall behavior of our rules is as with zero delivery costs.

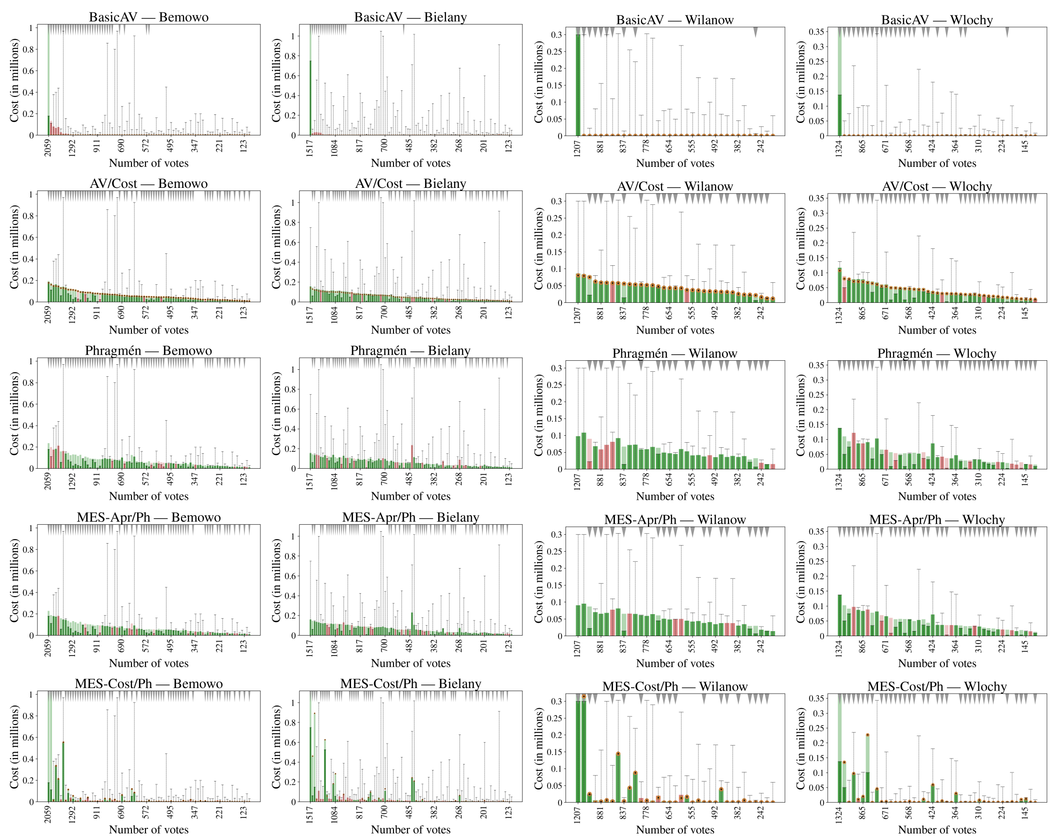

Figure 6: Strategy profiles after 10 000 iterations of our dynamics. Projects ordered by decreasing approval scores (x-axis). Final projects' costs shown by bars; the original costs shown as dashed lines. Red and green bars indicate, respectively, losing and winning projects; brighter parts are the increases over the original cost. The triangles at the top mark the originally winning projects. Brown circles denote the equilibrium costs (for rules for which we can compute it).

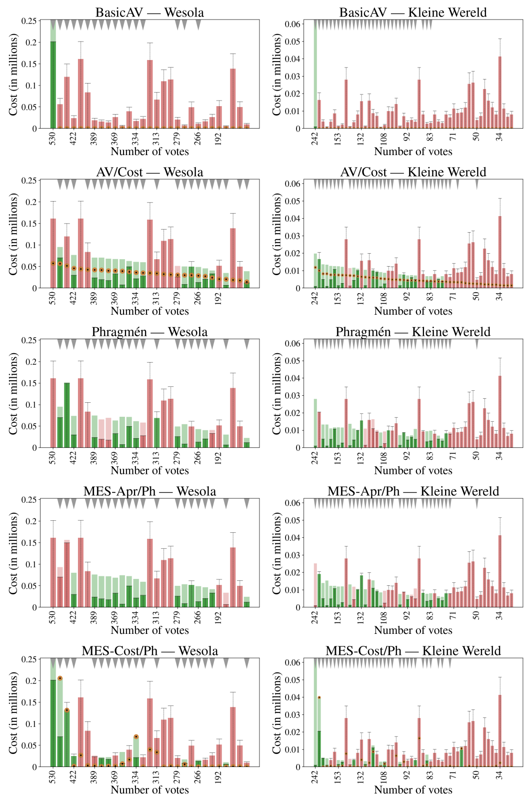

Figure 7: Strategy profiles after 10 000 iterations of our dynamics. for the case where each project has delivery cost equal to 80% of its original cost in the instance. Projects ordered by decreasing approval scores (x-axis). Final projects' costs shown by bars; the original costs shown as dashed lines. Red and green bars indicate, respectively, losing and winning projects; brighter parts are the increases over the original cost. The triangles at the top mark the originally winning projects. Brown circles denote the equilibrium costs (for rules for which we can compute it).